American Journal of Science, Engineering and Technology
2021; 6(4): 89-93
http://www.sciencepublishinggroup.com/j/ajset
doi: 10.11648/j.ajset.20210604.11
ISSN: 2578-8345 (Print); ISSN: 2578-8353 (Online)

Quantum Field Theory on Noncommutative Curved Space-times and Noncommutative Gravity

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To cite this article:
Subhamoy Singha Roy. Quantum Field Theory on Noncommutative Curved Space-times and Noncommutative Gravity. *American Journal of Science, Engineering and Technology*. Vol. 6, No. 4, 2021, pp. 89-93. doi: 10.11648/j.ajset.20210604.11

Received: August 13, 2021; **Accepted**: August 27, 2021; **Published**: October 5, 2021

Abstract: In the noncommutative gauge-theoretical formulation of Langman and Szabo, apparently it appears that the torsion generated there is a generalized one i.e. it may containvector, axial vector and tensor components. However, when we transcribe the noncommutative gauge theory in terms of the Maxwell gauge theory using the Seiberg-Wittencorrespondence, we have noted that upto the first order in the noncommutative parameter, this effectively can be taken to induce a change in chiral anomaly and hence theassociated torsion should be an axial vector one. The noncommutative gauge symmetriesgive a very natural and explicit realizations of the mixing of space-time and internal symmetries which is a characteristic feature of the conventional gauge theory of gravity. Thegauge fields of the dimensionally reduced noncommutative Yang-Mills theory map ontoaWeitzenböck space time and a teleparallel theory of gravity arises as the zero curvature reduction of a Poincare gauge theory which induces an Einstein-Cartan space-timecharacterized by connections with both nonvanishing torsion and curvature. However, theteleparallelism equivalent of general relativity involves all the components of torsion. Thechiral anomaly in the Einstein-Cartan space U_4 is characterized by the topological invariants like Pontryagin density as well as the Nieh-Yan density when the latter term involvesthe length scale governed by the measure of noncommutativity of space points. It is shownthat we have discussed the equivalence of this formalism with noncommutative U(1) Yang Mills theory.

Keywords: Noncommutative Space-Time, Teleparallel Gravity, Berry Phase

1. Introduction

A local field theory can be obtained by restricting noncommutative Yang-Mills fields such thatthe symmetry group contains diffeomorphism invariance. Noncommutative gauge symmetriesgive a very natural and explicit realizations of the mixing of space-time and internal symmetrieswhich is a characteristic feature of the conventional gauge theory of gravity. The gauge fields ofthe dimensionally reduced noncommutative Yang-Mills theory map onto a Weitzenböck spacetime and a teleparallel theory of gravity arises as the zero curvature reduction of a Poincaregauge theory which induces an Einstein-Cartan space-time characterized by connections withboth nonvanishing torsion and curvature. It has been sharp out that in the manifold $M_4 \times Z_2$ where the discrete space corresponds to a direction-vector, Z_2 symmetry breaking leads tochiral anomaly which is responsible for the topological origin of mass and describes electroweaktheory in a consistent way. Thus as in the case of a noncommutative manifold where Z_2 isa two-point space there appears to be a connection between gravity and electroweak theoryin this formalism this is achieved through the realization of chiral anomaly and torsion. Herewe shall study the noncommutative space-time having the structure $M_4 \times Z_2$ where Z_2 isnot a two point space but describes a 'direction-vector' attached to a space-time point. It isshown that this also leads to a torsioned space-time such that general relativity is describedas a teleparallel gravity. However unlike noncommutative Yang-Mills theory which involvesthe symplectic area preserving diffeomorphism, here we have deformation of the symplectic structure. Indeed this deformation of the symplectic structure is related to the Berry phasewhich is associated with chiral anomaly and this leads to a torsioned space-time.

In section 2 we shall discuss teleparallel gravity in the framework of this geometry and equivalence with theformulation of noncommutative gauge theory.

2. NoncommutativeGauge Theory, Chiral Anomaly and Torsion

Langmann and Szabo [1] have shown how noncommutative $U(1)$ Yang-Mills theory on the space $R^n \times R^n$ can generate a theory of gravitation on R^n . Let G^{AB} be a flat metric on R^{2n} . Thestandard action for Yang-Mills theory is defined by

$$S = \frac{1}{2}\int_{R^{2n}} d^{2n}\xi \sqrt{\det G} G^{AA'} G^{BB'} \hat{F}_{AB} * \hat{F}_{A'B'}(\xi) \tag{1}$$

where ξ denotes local coordinates of R^{2n} given by $\xi = (x^\mu, y^a)$ where $\mu, a = 1,2.......,n$. Thenoncommutative field strength is given by

$$\hat{F}_{AB} = \partial_A \hat{A}_B - \partial_B \hat{A}_A + e[\hat{A}_A * \hat{A}_B - \hat{A}_B * \hat{A}_A] \tag{2}$$

For completeness, we recapitulate here some of the crucial features of their observation. Thenoncommutativity parameters are taken to be of the block form

$$S = \frac{1}{2}\int_{R^{2n}} d^{2n}\xi W^{AA'BB'}(\xi) \hat{F}_{AB} * \hat{F}_{A'B'}(\xi) \tag{3}$$

For completeness, we recapitulate here some of the crucial features of their observation. Thenoncommutativity parameters are taken to be of the block form

$$\theta^{AB} = \begin{bmatrix} \theta^{\mu\nu} & \theta^{\mu b} \\ \theta^{\nu a} & \theta^{ab} \end{bmatrix} \text{with } \theta^{\mu\nu} = \theta^{ab} = 0 \tag{4}$$

The flat metric of R^{2n} is taken to be

$$g^{AB} = \begin{bmatrix} \eta^{\mu\nu} & \eta^{\mu b} \\ \eta^{\nu a} & \eta^{ab} \end{bmatrix} \text{with } \theta^{\mu\nu} = \theta^{ab} = 0 \tag{5}$$

The linear subspace of smooth function α on R^{2n} are taken to be linear in the coordinatesy

$$\alpha(\xi) = \alpha_a(x) y^a \tag{6}$$

We now define

$$g = Vect(R^n)$$

$$X_\alpha = -\theta^{\mu a}\alpha_a \frac{\partial}{\partial x^\mu} \tag{7}$$

so that we can have a representation of the Lie algebra

$$[X_\alpha, X_\beta] = X_{[\alpha,\beta]*}, \forall \alpha, \beta \in g \tag{8}$$

Here, g can be identified with the Lie algebra of connected difeomorphism of R^n . Defining

$$f^1(\xi) = f_a(x) y^a \text{ and } g^0(\xi) = g(x) \tag{9}$$

we have the star product rule

$$f^1 * g^0(\xi) = g(x) f_a(x) y^a - \frac{1}{2}\theta^{\mu a} f_a(x) \partial_\mu g(x) \tag{10}$$

From this we can consider the gauge transformation rule

$$\partial\phi = -\theta^{\mu a}\alpha_a \partial_\mu \phi \tag{11}$$

Under a global transformation $x^\mu \to x^\mu + e^\mu$, the scalar fields transform infinitesimally as

$$\phi(x) \to \phi(x) + e^\mu \partial_\mu \phi(x) \tag{12}$$

Since

$$\partial_\mu \phi(x) = -(\theta^{-1})_{a\mu}[y^a, \phi] * (x) \tag{13}$$

we can identify y^a with the holonomic derivative generators $-\theta^{\mu a}\partial_\mu$ of the n-dimensional translational group T of R_x^n . This helps us to write the covariant derivative

$$\nabla_\mu = \partial_\mu - e\omega_{\mu a}\theta^{\nu a}\partial_\nu \tag{14}$$

where $\omega_{\mu a}$ are gauge fields corresponding to the gauging of the translation group. Now defining curvature of the gauge field

$$\hat{F}_{\mu\nu}(\xi) = \Omega_{\mu\nu a} y^a \tag{15}$$

where

$$\Omega_{\mu\nu a} = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + e\theta^{\lambda b}(\omega_{\nu b}\partial_\lambda \omega_{\mu a} - \omega_{\mu b}\partial_\lambda \omega_{\nu a}) \tag{16}$$

and replacing y^a by $-\theta^{\nu a}\partial_\nu$ we note that the covariant derivative ∇_μ defines a nonholonomicbasis of the tangent bundle with nonholonomicity given by the field strength tensor. Thecommutator of covariant derivatives is given by

$$(\nabla_\mu, \nabla_\nu) = T^\lambda_{\mu\nu}\nabla_\lambda \tag{17}$$

This commutation relation identifies $T^\lambda_{\mu\nu}$ or equivalently noncommutative gauge field strength $\Omega_{\mu\nu a}$ as the torsion tensor fields of vacuum space-time induced by the presence of a gravitationalfield. This induces a teleparallel structure of space-time through the Weitzenböckconnectionwith nonvanishing torsion and vanishing curvature. Indeed the torsion $T^\lambda_{\mu\nu}$ measures the noncommutativity of displacement of points in the flat space-time R_x^n . It is dual to the Riemanncurvature tensor which measures noncommutativity of vector displacements in a curved space-times. Now we consider the equivalence of this formalism of gravity from noncommutative $U(1)$ Yang-Mills field with that attained from the noncommutative geometry $M_4 \times Z_2$ where spacetime coordinates are extended by $SL(2,C)$ gauge fields and torsion is obtained from chiralanomaly. Indeed we can identify the position and momentum operators as

$$\frac{Q_\mu}{\omega} = -i(\frac{\partial}{\partial p_\mu} + A_\mu)$$

$$\frac{P_\mu}{\omega} = -i(\frac{\partial}{\partial q_\mu} + A_\mu) \tag{18}$$

where ω is the dimensionless variable, $\omega = \frac{\hbar}{lmc}$. Here q_μ is the space-time point in Minkowski space M_4 and p_μ its conjugate representing the momentum variable. We have pointed out thatthe gauge theoretical extension of space-time and momentum coordinate given by eqn.(18) effectively deforms the symplectic structure. Indeed from equation (18) we note that the symplectic form of the phase space will now be given b

$$\Omega = \frac{1}{2} g^{ij} dp_i \Lambda dq_j \tag{19}$$

thesymplectic form is given by eqn.(19) where the matrix Δ^{ij} is associated with the gaugefield strength. Indeed by replacing the indices i, j by μ, ν, we may associate it with the fieldstrength $F_{\mu\nu}(\vec{\xi})$ with $\vec{\xi} = (\vec{x}, \vec{P})$. Now following eqn.(15) we write

$$F_{\mu\nu}(\vec{\xi}) = \Omega_{\mu\nu a} \theta^{\lambda a} \partial_\lambda \tag{20}$$

where $\Omega_{\mu\nu a}$ is the field strength associated with the gauge field $\omega_{\mu a}$ given by eqn.(16). It isnoted that $\theta^{\nu a}$ here effectively corresponds to the matrix j^{ij}

$$g^{ij} = j^{ij} + \hbar \Delta^{ij} \tag{21}$$

in eqn.(21) and represents the usual symplectic structure. To trace the nature of the gauge field $\omega_{\mu a}$, we decompose it in the form

$$\omega_{\mu a}(x) = W_\mu \overline{\theta}^{a\nu} x_\nu \tag{22}$$

Where

$$\overline{\theta}^{a\nu} = i\mu \varepsilon^{a\nu\lambda} \frac{x_\lambda}{r^3} \tag{23}$$

$\varepsilon^{a\nu\lambda}$ being the fundamental Levi-Civita tensor. In eqn. (23) μappears as the monopole strength. It may be noted that $\overline{\theta}^{a\nu}$ here is essentially related to the noncommutativity of momentum space. Indeed when we consider that the symplectic matrix is not a constant one but dependent onxorp, it violates the associativity property. The failure of the Jacobi identity

$$[x^i,[x^j,x^k]] + [x^j,[x^k,x^i]] + [x^k,[x^i,x^j]] \neq 0 \tag{24}$$

implies the presence of the dual monopole which is analogous to the failure of the Jacobiidentity between momentum implying the presence of the Dirac monopole [2]. Thus replacing the constant symplectic matrix by the gauge field strength $F_{\mu\nu}$ effectively implies that the corresponding gauge field is operative in the presence of a magnetic monopole. This essentially indicates that the gauge field W_μ in eqn. (22) should be a non-Abelian one. This follows from the fact that we can assciate the monopole strength μ with the Pontryagin index q associated with the θ - term in non-Abelian gauge field. In fact, we have the relation [3]

$$q = 2\mu = \frac{1}{16\pi^2} Tr \int *F_{\mu\nu} F_{\mu\nu} d^4 x \tag{25}$$

The hidden Abelian gauge field in a non-Abelian gauge theory associated with the topological θ-term $*F_{\mu\nu}F_{\mu\nu}$ in the Lagrangian may be viewed as if in the gauge orbit space, the position of a particle is indicated by A (non-Abelian gauge potential) moving in the space U of non-Abelian gauge potentials under the influence of an Abelian electromagnetic potential [4, 5]. Wecan write

$$A = g^{-1} dg + g^{-1} a g \tag{26}$$

where $A = A_\mu dx^\mu, a = a_\mu dx^\mu$ and $g(x)$ is a matrix associated with the gauge transformation. The space of gauge orbits U/G where Gdenotes the space of local transformations of $g(x)$ consists of the point $a(x)$. Recalling that $\Pi_3(G) = Z$ for all simple non-Abelian groups G and $\Pi_2(G) = 0, \Pi_n(U) = 0$ for all n, we have

$$\Pi_n(U/G) = \Pi_{n-1}(G), n \rangle 1 \tag{27}$$

That means in 3 + 1 dimension

$$\Pi_1(U/G) = \Pi_0(G) = \Pi_3(G) = Z \tag{28}$$

This equality $\Pi_0(G) = \Pi_3(G) = Z$ follows from the condition that the gauge transformation $g(x)$ approaches constant independent of the direction of x as $x \to \alpha$. Thus U/G is multiplyconnected and has the topology of a ring and the corresponding field strength corresponds to a vortex line which is topologically equivalent to a magnetic flux line. Now we note that when a particle moves in the presence of a magnetic monopole, the angular momentum is given by

$$\vec{J} = \vec{r} \times \vec{p} - \mu \hat{r}, \mu = \frac{1}{2}, \pm 1, \pm \frac{3}{2}, ... \tag{29}$$

In case $|\mu| = \frac{1}{2}$, we can associate this with a spinor and in view of this we consider W_μ as an $SU(2)$ gauge field. In 3+1 dimension, the group structure can be generalized to $SL(2,C)$ SL. Thuswe find that the torsion generated by the noncommutative gauge field theory may be consideredto be equivalent to the torsion generated by the $SL(2,C)$ gauge theory in noncommutativespace $M_4 \times Z_2$.

We have observed in the previous section that the noncommutative manifold $M_4 \times Z_2$ induces torsion through chiral anomaly which is related to the Berry Phase associated with the deformation of the symplectic structure. The formulation of noncommutativeU(1) Yang –Mills theory also implicitly induces the change in chiral anomaly. However, in this case we have area preserving diffeomorphism. Indeed, the association of the gauge field $\omega_{\mu a}$ with the gaugefield W_μ in presence of a monopole related to the θ-term in non-Abelian gauge field theory effectively relates the torsion term associated with the noncommutative $U(1)$ gauge field theorywith the chiral anomaly. In fact, the topological term $Tr * F_{\mu\nu} F_{\mu\nu}$ effectively corresponds to thedivergence of the axial vector current and hence we may view that the torsion is of axial vectorin nature. This can be shown in a more explicit form from the following considerations. Thenoncommutative field strength constructed from the potential $\hat{A}_\mu$ is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g[\hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu] \qquad (30)$$

when we have the Lagrangian

$$\hat{L} = -\frac{1}{4} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \qquad (31)$$

In terms of the conventional Maxwell term

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (32)$$

it turns out that to first order in $\theta^{\alpha\beta}$ we can express $\hat{A}_\mu$ and $\hat{F}_{\mu\nu}$ as follows [6]

$$\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) \qquad (33)$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu} \qquad (34)$$

where g is absorbed in θ. The tensor $\theta^{\alpha\beta}$ is associated with the star product which is defined as

$$(f * g)(x) = e^{\frac{i}{2} \theta^{\alpha\beta} \partial_\alpha \partial_\beta} f(g)\, g(x')\Big|_{x=x'} \qquad (35)$$

Apart from the total derivative term, the Lagrangian $\hat{L}$ is given by

$$\hat{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu} + O(\theta^2) \qquad (36)$$

It is well known that star product effectively involves a background magnetic field [7] and sothe second and the third terms in equations (34) and (36) correspond to the interaction of thisbackground field with the Maxwell field. When we consider the interaction of the chiral current with the noncommutative gauge field having the field strength to the first order in θ, we have

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu} = F_{\mu\nu} + \tilde{F}_{\mu\nu} \qquad (37)$$

We note that the chiral anomaly will be modified as

$$\frac{1}{8\pi^2} Tr \int [{}^*F_{\mu\nu} F_{\mu\nu} + {}^*F_{\mu\nu} \tilde{F}_{\mu\nu} + {}^*\tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu}] \qquad (38)$$

This implies that the induced background magnetic field associated with the star producteffectively changes the chiral anomaly. In the noncommutative gauge-theoretical formulation of Langman and Szabo, apparently it appears that the torsion generated there is a generalized one i.e. it may contain vector, axial vector and tensor components. However, when we transcribe the noncommutative gauge theory in terms of the Maxwell gauge theory using the Seiberg-Witten correspondence, we have noted that upto the first order in the noncommutativeparameter θ, this effectively can be taken to induce a change in chiral anomaly and hence the associated torsion should be an axial vector one. However, the teleparallelism equivalent of general relativity involves all the components of torsion. In fact, if we denote the Lagrangian in teleparallel gravity as [8]

$$L_T = -\frac{1}{2l^2} T^\alpha \wedge^* (-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha) \qquad (39)$$

where ${}^{(1)}T_\alpha$, ${}^{(2)}T_\alpha$, ${}^{(3)}T_\alpha$ corresponds to tensor, vector (trace) and axial vector components of torsion, its equivalence with general relativity arises from the geometrical identity

$$-\frac{1}{2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2} R^{\{\ \}\alpha\beta} \wedge \eta_{\alpha\beta} + l^2 L_T = d(e^\alpha \wedge^* T_\alpha) \qquad (40)$$

where $R^{\alpha\beta}$ is the curvature associated with the Einstein-Cartan space-time, $R^{\{\ \}\alpha\beta}$ is the curvature in Riemannian space-time and $\eta_{\alpha\beta} = {}^*(e_\alpha \wedge e_\beta)$. From this we note that in Weitzenböckspace-time with vanishing Riemann-Cartan space curvature $R^{\alpha\beta} = 0$ the Lagrangian L_T isupto a boundary term equivalent to the Einstein-Hilbert Lagrangian. As the teleparallelLagrangian involves all the components of torsion with specific coefficients, only the axial vectorcomponent of torsion can contribute if by any constraint the contributions of the vector andtensor components in the action cancel.

3. Discussion

In our present framework, we observe that Z_2 symmetry breaking leads to chiral anomaly. In recent paper [9-11] it has been shown that chiral anomaly gives rise to the topologicalorigin of mass. Besides chiral anomaly is associated with the nonvanishing value of the Barryphase factor [12, 13]. This is linked up with the conserved charge through Direc quantizationcondition. So we observed that the symmetry breaking in the noncommutative space- timegiven by $M_4 \times Z_2$ when the discreates space is considered as the internal space not only generatemass but also a locally defined conserved charge. Recently it has been pointed out that the N-Ydensity may be taken to arise from the noncommutativity of space when the space-time manifoldis $M_4 \times Z_2$, we note that gravitational constant may be considered to be a manifestation of thenoncommutative geometry. It may be remarked that in the noncommutative $U(1)$ Yang-Millsfield theoretical formulation also the gravitational constant is found to be associated with thegauge coupling constant and noncommutativity scale [1].

4. Conclusion

Finally We may add that in flat space as the manifold $M_4 \times Z_2$ is found to be associatedwith the quantization of a fermion when the Z_2 symmetry breaking leads to chiral anomalywhich is responsible for the origin of mass [14], it is very natural to link up gravitation withthe noncommutativity of space and in this way the association of gravitation with quantummechanics becomes relevant [15, 16]. Thus the noncommutative geometry paves the path to havea reconciliation of general relativity and quantum mechanics.

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